

Asymmetric equilibria in dynamic two-sided matching markets with independent preferences

Kimmo Eriksson · Jonas Sjöstrand ·
Pontus Strimling

Accepted: 16 March 2007 / Published online: 12 April 2007
© Springer-Verlag 2007

Abstract A fundamental fact in two-sided matching is that if a market allows several stable outcomes, then one is optimal for all men in the sense that no man would prefer another stable outcome. We study a related phenomenon of asymmetric equilibria in a dynamic market where agents enter and search for a mate for at most n rounds before exiting again. Assuming independent preferences, we find that this game has multiple equilibria, some of which are highly asymmetric between sexes. We also investigate how the set of equilibria depends on a sex difference in the outside option of not being mated at all.

Keywords Mutual mate choice · Strategic mating · Independent preferences · Equilibrium · Multiple equilibria · Asymmetric equilibria

JEL Classification Numbers Primary: 06A07 · Secondary: 05E10

1 Introduction

Is she good enough for me? If I leave him will I find someone better? Will that one want me? Among humans as well as among animals these questions lead up to the most important question there is, at least from a genetic point of view: Should I mate with this one or not? Due to its importance, mate search has become a cross disciplinary line of study, comprising both what tactics the individuals use and what effects these

K. Eriksson (✉) · P. Strimling
Mälardalen University, IMA, Box 883, 721 23 Västerås, Sweden
e-mail: kimmo.eriksson@mdh.se

J. Sjöstrand
Royal Institute of Technology, Stockholm, Sweden

tactics have on the overall population. In this paper we take a closer look at asymmetric outcomes, where one sex has a somehow better situation than the other.

For matching markets that are static in the sense that the set of agents is unchanging, game theory has developed the powerful theory of stable matching.¹ In particular, it is known that there exists at least one stable matching (Gale and Shapley 1962), and that if several stable matchings exist then among them is one that is optimal for all agents of one side, say men, and the worst possible stable outcome for all agents of the other side (Knuth 1976; Demange and Gale 1985; Roth and Sotomayor 1990). Hence asymmetric outcomes are well known in this theory.

In the present paper, we keep this idea in mind while departing from stable matching theory. Mate search situations and other decentralized matching markets usually go beyond the assumptions of stable matching theory in that they are dynamic in the set of agents. To begin with, agents are usually assumed to leave the market when they mate. On the human mating market it is also an important feature that young people enter the market every year, and old people leave the market even if not mated. Mathematical models for such dynamic matching markets, usually called “mutual mate choice models” in biology exist in a tradition separate from stable matching theory (Parker 1983; Real 1991; Johnstone et al. 1996; Johnstone 1997; Alpern and Reyniers 1999, 2005; for a review see Alpern et al. 2005). Whereas any preference structure is regarded as viable input to a stable matching model, the biological tradition usually works only with common preferences (meaning that all males have identical preference relations over the females and vice versa). An exception is a study of Alpern and Reyniers (1999) with an assumption of homotypic preferences, where individuals prefer mates similar to themselves. In our model, we will make an even clearer break with the biological tradition and instead work with preferences that are completely independent between agents. Although there are probably no examples of completely independent preferences in mate search, it seems to us to be the natural baseline model.

Asymmetric outcomes have been studied within the biological tradition by Johnstone et al. (1996). However, they introduced sex differences exogenously in the game, such as different costs of choice or a sex ratio not equal to 1:1. The main objective of the present paper is to study whether sex-distinct mating strategies, yielding asymmetric outcomes, can emerge *even when the game is perfectly symmetric*. For homotypic preferences, such asymmetric equilibria are ruled out by definition, since the outcome is always equally good for the male and the female in any pair, hence symmetric. In the case of common preferences, asymmetric equilibria of symmetric mating games are possible, but neither Johnstone (1997) nor Alpern and Reyniers (2005) find any. In fact, Alpern and Reyniers (2005) prove that in their symmetric mating model with $n = 2$ periods, no asymmetric equilibrium is possible. They indicate that their argument might be extendable to any number of periods. So, it seems likely that emergence of asymmetric outcomes is not possible in these particular models. In the light of stable matching theory this is perhaps not so surprising: common preferences give only one stable matching, and hence we would see no asymmetric

¹ A matching is stable if there is no man and woman who both prefer each other to their current partners.

outcomes in a static market either. We will show that in our dynamic model with independent preferences:

- *heavily asymmetric equilibria may exist even if the game is perfectly symmetric, and even if there are only $n = 2$ periods.*

We will also investigate the effect of outside options, another phenomenon that is well known to economists but absent from the biological modeling tradition. In a matching market, the outside option is the option of staying single. Evolutionary biology and evolutionary psychology maintain that among mammals in general, including humans, females have more to gain from having a long-term reproductive partner (Geary 2000, 2005), and hence females have more to lose if they are alone. For humans this finding is supported by American survey results (Wood et al. 1989) showing that single women report the lowest level of well-being, and married women the highest (with single men and married men in the middle, in that order); see also a literature review of Coombs (1991), although the results have been contested (Stack and Eshleman 1998). More to the point, stated preference for married life to single life is a strong predictor for marriage among women but not for men (Sassler and Schoen 1999), suggesting a strategic difference between the sexes, that it the preference is more important to act upon for females. It is obvious that gender differences in the value of mating need not be biologically founded but can arise from cultural practices (e.g. Jane Austen's novels describe a culture where not marrying is particularly unappealing to women, but another culture might traumatize men's loneliness more) the important thing here is that there is a difference in outside option. These investigations also say nothing about the size of the difference, it might be that compared to the difference in preference over mates it is very small. In this paper, we investigate how such a sex difference in outside option affects the equilibria of our mating game. Our two main findings are that

- *already a quite small sex difference in the utility of staying unmated can change the set of equilibria so that all equilibria are much better for males;*
- *if the outside option is generally not so bad, we find a collapse to a single equilibrium (favoring males in the asymmetric case).*

Dynamic matching is much more difficult to analyze than static matching. To understand what is going on in the model we have had to rely on massive computations to find exact solutions for many given parameter values. Thanks to the simplicity of working with independent preferences we have managed to explain most of our general findings also by approximations amenable to mathematical analysis (assisted by computer algebra), but we believe this model is on the border of where general mathematical analysis is possible.

The paper is organized as follows. In Sect. 2, we define the model. In Sect. 3, we demonstrate the fundamental backward induction recurrence that governs optimal strategies for each sex in this model, and show how acceptance rates and equilibria are derived. In Sect. 4, we show the results of computational analysis of how the set of equilibria depends on the various parameters. In Sect. 5, we explain our findings by mathematical analysis. (Details are given in an appendix.) Finally, we discuss how our results relate to other preference structures.

2 The model

We approach mutual mate choice as a process of agents selecting mates based on preferences over possible mates, trying to solve an “optimal stopping” problem: agents have a time limit imposed as a maximum of n periods in which to observe potential mates, one per period. If two agents who meet in a period agree to mate, they leave the game (i.e. they “stop”).

Our model does not come directly from either of the traditions of stable matching or mutual mate choice; instead it draws upon the literature on the celebrated “secretary problem” and its two-sided extensions (cf. Ramsey and Szajowski 2005). Chow et al. (1964) solved a one-sided selection model known as the optimal-rank version of the secretary problem. This model was extended to a two-sided strategic model by Eriksson et al. (2006), where different agents were assumed to hold independent preferences. In this model all agents are always in the same period in their mate search, and the equilibrium was by necessity unique and symmetric. Here, we will change this model so that mating can occur across cohorts, making the influx of young agents available as mates to older agents. We also introduce parameters u^M , u^F representing how the two sexes value the outside option.

The basic features of our model can be summarized as follows:

- There are always equally many available agents in the game, and the number of males and females are equal.
- Each agent has n periods available for dating.
- In each period, all available males and females are randomly matched to each other.
- If both agents at a date accept to get mated they leave the game. An agent also leaves the game if he remains unmated after his last period.
- When an agent leaves the game, another agent of the same sex enters immediately. (The aim of the model is not to study the effects of variation in group size or sex distribution, and this is the simplest way of keeping those parameters static.)
- Different agents’ preferences are completely independent, except that for all males (resp. females) the option of ending up single is ranked $u^M n$ (resp. $u^F n$).
- All agents seek to optimize (i.e. minimize) the expected rank of their mate.

The last point is worth commenting on. In traditional biological models one think of agents choosing mates on the basis of biological fitness, hence the usual assumption of common preferences. When preferences differ between agents they must be operationalized by some measure, the rank. As usual in the “secretary problem” tradition, we assume that agents do not have any a priori knowledge of the distribution of those underlying desirable characteristics that are manifested in the rank.

2.1 Preferences and ranks

The most subtle part of our model is the definition of preferences and ranks. For the sake of simplicity, we will take the male perspective and reason about male agents; the analogous statements hold for females. Each agent has an *implicit rank order* of all females that he could possibly ever date, i.e. those females that are available when he

enters the game and those that will enter the game during the n periods in which he is possibly in the game himself. For each agent, this rank order is independently drawn from a uniform distribution of all possible permutations. Since a random choice determines who will date who in each period, it follows that from the viewpoint of any player entering period r , the rank of the next date relative the $r - 1$ partners already observed is a random variable drawn from a uniform distribution on the set of ranks from 1 to r .

We want to stress that agents do not observe the values of the implicit ranks, but can only compare agents they have seen. Therefore, an agent cannot make any informed decision on the first date: even the best date in the universe would make as good or bad impression as the worst date in the universe. Only later comparisons will reveal how good or bad the first date really was.

Independence of preferences also means that an agent cannot draw any conclusion about whether or not future partners will accept him from what past matched partners did.

We assume that each agent wants to minimize the expected rank of his mate among the n partners he would meet if he completed all n periods. Although the actual set of mates that an agent would meet in the remaining periods is not known, it follows from the assumptions of uniformity and independence that the *expected final mate-rank* is

$$\frac{n + 1}{r + 1} \rho \tag{1}$$

of a mate who is ranked ρ among the r partners observed up to period r (Eriksson et al. 2006).

2.2 Strategies and expected payoffs

A *strategy* in this two-sided secretary game is a rule that says for each period r whether to accept a date of observed rank ρ in this period.

Expected payoff of a given strategy depends on the strategy profile of all agents. Payoffs in our game is defined by the final mate-rank. Given a strategy profile, we can define R_r^i as the expected final mate-rank for a certain agent of sex i entering period r . Thus, agents want to minimize R_1 , the expected final rank at the start of the game. The following fundamental recurrence governs the expected final mate-rank when a player of sex i enters period r :

$$R_r^i = P[\text{mate}] \cdot \frac{n + 1}{r + 1} \cdot E[\rho|\text{mate}] + (1 - P[\text{mate}]) \cdot R_{r+1}^i. \tag{2}$$

As before in (1), ρ denotes the rank of the date in period r among the r partners you have seen. If one remains not mated after the last period, one obtains the “empty mate” which is ranked $u^M n$ and $u^F n$, respectively, for males and females. Thus, we put

$$R_{n+1}^i = u^i n \quad \text{for } i = M, F. \tag{3}$$

We will assume that $0 < u^F, u^M \leq 1$.

2.3 Steady state equilibria

The game has reached a steady state if among all available males (resp. females) the proportion currently being in a given period is constant. Since all available agents of the opposite sex are equally likely to be your next date, in a steady state the probability that a male will be accepted by the next female is always the same. We denote this probability by α^F , with α^M analogously defined.

By an *equilibrium* we will mean a strategy profile such that the game maintains a steady state, and such that every agent in each period optimizes the expected payoff given this steady state.

Since all males (resp. all females) have decision problems described by identical equations, all males (females) will have an identical strategy, up to unimportant tie-breaking. Thus we will conceive of strategy profiles as strategy *pairs* (s^F, s^M) .

Remark Computer simulations support that for any strategy pair a steady state is eventually reached, although it seems difficult to prove that convergence must necessarily occur (cyclic or simply chaotic behavior being alternatives that cannot be a priori ruled out).

3 Basic analysis of equilibrium behavior

Let s_r^M be the threshold defining the male strategy in period r , that is, the male accepts if the rank he observes in this period is at most s_r^M . This means that, given that he has reached period r , the probability that he will accept in period r is s_r^M/r . Hence, the probability of an agreement to mate is $P[\text{mate}] = \alpha^F s_r^M/r$. Given that the male accepts, the expected observed rank of his partner is $E[\rho|\text{mate}] = (s_r^M + 1)/2$. We can plug these expressions into the fundamental recurrence (2):

$$R_r^M = \alpha^F \frac{s_r^M}{r} \cdot \frac{n+1}{r+1} \cdot \frac{s_r^M + 1}{2} + \left(1 - \alpha^F \frac{s_r^M}{r}\right) \cdot R_{r+1}^M. \tag{4}$$

An agent should accept in period r if and only if the expected final mate-rank if he mates now is less than or equal to the expected final mate-rank if he does not mate. This equilibrium condition takes the following mathematical form.

Equilibrium condition:
$$s_r^M = \left\lfloor \frac{r+1}{n+1} \cdot R_{r+1}^M \right\rfloor, \quad r = 1, \dots, n \tag{5}$$

Using (4) and (5) we can compute the exact equilibrium thresholds for the males given values of the number n of periods, the boundary condition $R_{n+1}^M = u^M n$, and the female average acceptance rate α^F . Figure 1 illustrates the optimal strategy for one particular choice of parameter values. The value of u^M determines the threshold in the last period, while the value of α^F determines the rate by which the thresholds are lowered in earlier periods.

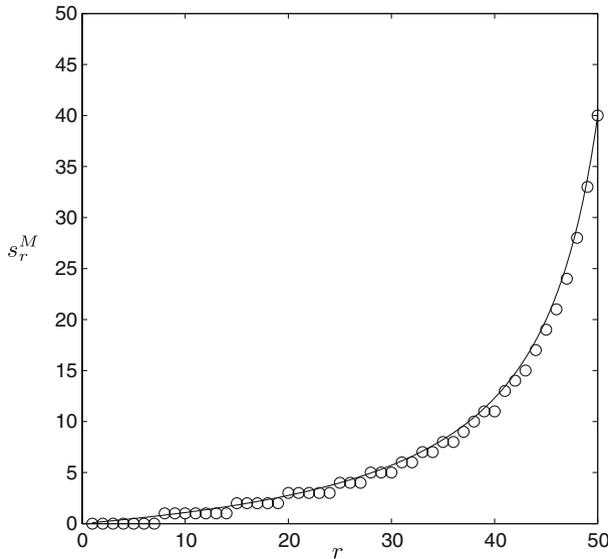


Fig. 1 The circles show the exact optimal male strategy for $n = 50$, $u^M = 0.8$, and $\alpha^F = 0.4$. The curve shows an approximate optimal strategy to be derived in Sect. 5

3.1 Acceptance rates

Given the equilibrium strategy of males, we shall now compute what average acceptance probability α_{opt}^M this entails. Given that a male arrives in period r , the probability that he will accept the date in this period is s_r^M/r , so we just need to weight this number with the probability P_r^M that a random male will currently be in period r :

$$\alpha_{opt}^M = \sum_{r=1}^n P_r^M s_r^M / r. \tag{6}$$

By the steady state assumption, the probability P_r^M can be expressed in terms of the probability p_r^M that a male will reach period r :

$$P_r^M = \frac{p_r^M}{\sum_{i=1}^n p_i^M}. \tag{7}$$

The probability is $1 - \alpha^F s_i^M / i$ that a male that has arrived at period i will *not* mate in this period. Hence the probability p_r^M that this male will reach period r is

$$p_r^M = \prod_{i=1}^{r-1} (1 - \alpha^F s_i^M / i). \tag{8}$$

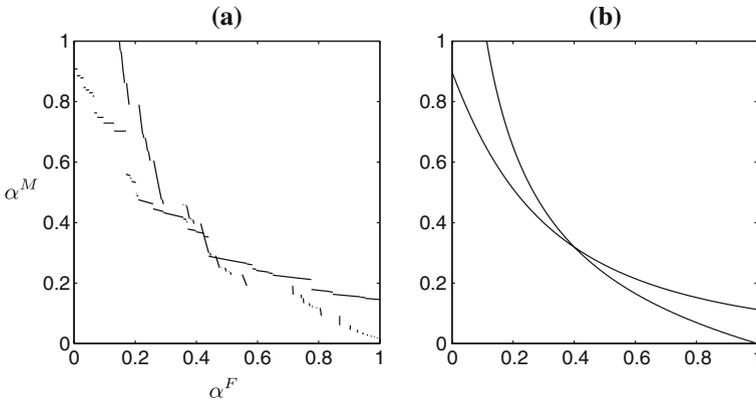


Fig. 2 **a** *Left* Optimal response acceptance probabilities for both sexes. Here $n = 11, u^F = 1.0, u^M = 0.9$. *Crossings* signify equilibria. **b** *Right* In Sect. 5, we will derive approximate optimal responses. The figure shows this approximation, according to Eq. (12), for the same parameter values as in the *left* figure

3.2 Equilibria

In the symmetrical way we can compute the optimal female response s^F to a male acceptance rate of α^M , and derive the corresponding female acceptance rate α_{opt}^F . If we plot the best male response α_{opt}^M for all values of α^F between 0 and 1, and in the same diagram plot the best female response α_{opt}^F for all values of α^M , then we obtain Fig. 2a. Since equilibria are pairs of strategies (s^F, s^M) that are best responses to each other, they correspond exactly to the intersections of the graphs.

The striking feature of Fig. 2a is that the graphs are discontinuous, consisting of many smooth segments. The simple explanation is that each segment corresponds to a strategy s^M (resp. s^F), i.e. a set of thresholds. Since there is only a finite number of possible thresholds, there must be a discrete set of points at which there is a switch of optimal strategy. Segments end when the optimal strategy changes, which happens when some threshold is changed one step due to the floor function in (5).

It is possible to show that the number of segments grows quadratically in n . Intuitively, the number of segments is the difference between the sum of thresholds at $\alpha = 0$ and $\alpha = 1$, respectively; these sums can be estimated using the techniques developed in Sect. 5. We omit the details.

4 Computational analysis of equilibria

Since equilibria show as intersection points of segments in Fig. 2a, we will think of equilibria as pairs (α^F, α^M) , although this is only an outcome of the underlying strategy pair (s^F, s^M) .

All these equilibria are locally belief-enforcing in the following sense. If males have a slightly wrong belief about α^F , but close enough to play the optimal strategy, and similarly for females, then they will continue to play the equilibrium strategies while learning the accurate values of the acceptance rates of the other sex.

Furthermore, it is easy to see that the two function graphs in Fig. 2a are oriented relative each other in such a way that an iterative best-response dynamic will converge to an equilibrium from any initial beliefs: the graph of α_{opt}^M starts to the left of α_{opt}^F and ends above it. We have used this dynamic to compute all equilibria for various parameter settings, using iterative methods. Thanks to the discrete character of the underlying strategy space, it is possible to ascertain that the search is exhaustive.

4.1 Equilibria in the symmetric case: $u^M = u^F$

Suppose that $u^M = u^F$, so that our model of mutual mate choice is perfectly symmetric. Figure 3 (top) shows that there can indeed exist many different equilibria, with different strategies and different outcomes for the two sexes. Figure 3 (bottom) shows that the number of equilibria varies with n in a seemingly chaotic way.

Consequently, one must ask how large the sex difference can be in equilibrium. Figure 4 shows both the maximal and minimal equilibrium values of α^F . By symmetry, the corresponding curves for α^M are identical, with the maximal value of α^F attained simultaneously with the minimal value of α^M , and vice versa. The figure

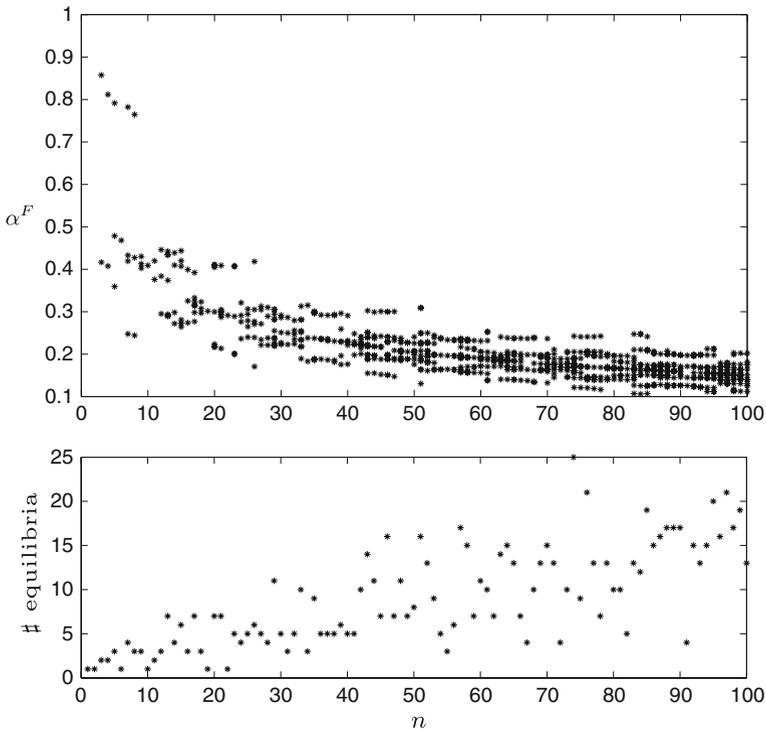


Fig. 3 The α^F values (and simultaneously the α^M values) of all equilibria (top) and the number of equilibria (bottom). Here $u^F = u^M = 1$

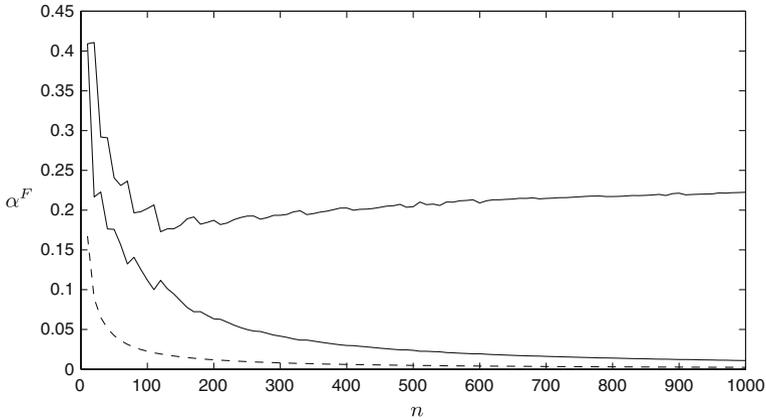


Fig. 4 The *solid curves* show the minimal respectively maximal α^F value at an equilibrium. The *dashed curve* is the product of the other curves. Here $u^F = u^M = 1$

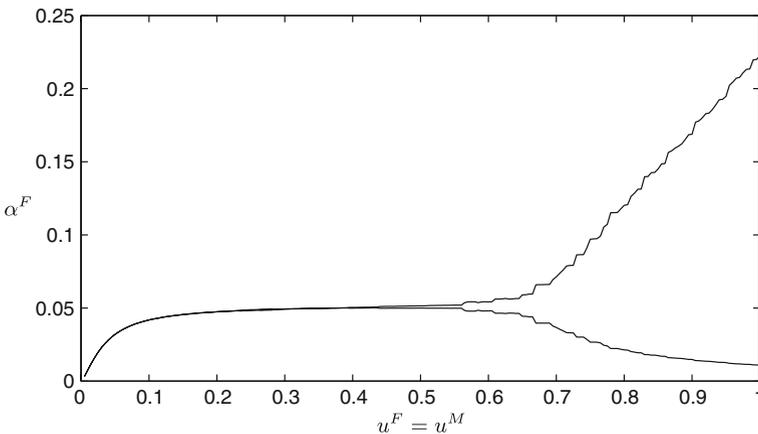


Fig. 5 The minimal and maximal α^F values at an equilibrium plotted versus $u^F = u^M$, with $n = 1,000$

indicates that the sex difference can become very large for large n . The maximal value of the acceptance rate α^F seems to approach a constant, while the minimal value tends to zero. The product of the maximal and minimal acceptance rates is shown by the smooth dashed curve, which seems to be proportional to $1/n$. In fact, we shall later explain why the product $\alpha^F \alpha^M$ in any equilibrium is proportional to $1/n$.

Another question is how the behavior we have seen above depends on the value of $u^F = u^M$, i.e. the outside option of not mating at all. In Fig. 5, we see that the equilibrium is essentially unique when the cost of not mating is sufficiently small, with a threshold value somewhere around 0.6. Above this threshold, the possible asymmetry of acceptance rates in equilibrium increases fast.

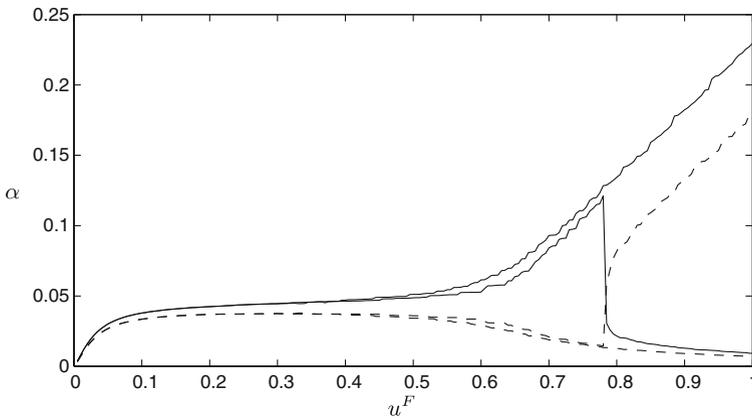


Fig. 6 The minimal and maximal α values at an equilibrium plotted versus u^F . Solid: α^F . Dashed: α^M . We have $n = 1500$ and $u^M = 0.9u^F$

4.2 Equilibria in the asymmetric case: $u^M < u^F$

As discussed in the introduction, a reasonable assumption is that the payoff to leave the game unmated is different for males and females. Figure 6 shows, for a fixed large n , the effect of a slight sex difference in outside options favoring males: $u^M/u^F = 0.9$. We see that for values of u^F less than about 0.8, even in the worst possible equilibrium for males they maintain a lower acceptance rate than women, i.e. they are choosier. As we will explain later, the outcome in terms of how preferable a partner players can expect to obtain is directly and positively linked to their choosiness in equilibrium. Hence, when $u^F < 0.8$ we see that all equilibria are better for males. Then, suddenly an equilibrium emerges that is better for the females.

4.3 Summary of computational findings

When the costs of not being mated are not too bad, say below 0.6, then there is a unique equilibrium. In a symmetric setting this unique equilibrium is symmetric, but if males are better off not mated than females then the equilibrium is better for males.

When the costs are higher than about 0.6, several very different equilibria exist. In the best equilibrium for males, the female acceptance rate is approximately constant with respect to n while the male acceptance rate is inversely proportional to n . In all equilibria the product $\alpha^M \alpha^F$ is inversely proportional to n .

For asymmetric games favoring males, all equilibria tend to be better for males. However, if the costs are high enough, then even in an asymmetrical setting there exist equilibria favoring the sex that is worst off when not mated. In the following, we shall analytically explain these features of the model.

5 Approximative mathematical analysis

The recurrence (4) does not allow an explicit closed form solution. In this section, we shall derive an approximative solution for a fixed male belief about the value of α^F . Ignoring the floor function in (5) and assuming that n , r and s_r^M are large, the recurrence (4) reduces to

$$R_r^M \approx R_{r+1}^M - \alpha^F \frac{(R_{r+1}^M)^2}{2n}. \quad (9)$$

Approximating the difference by the derivative yields a simple differential equation with an explicit unique solution: $R_r^M \approx 2n/(C - \alpha^F r)$, for some constant C to be determined by the boundary condition. After the last period, unmated agents obtain the empty mate ranked $u^M n$. Hence $2n/(C - \alpha^F(n+1)) = u^M n$, which implies that $C = \alpha^F(n+1 + \gamma^F) \approx \alpha^F(n + \gamma^F)$ where we have let $\gamma^F = 2/(u^M \alpha^F)$. We have now derived a simple approximation of the expected final mate-rank when entering period r :

$$R_r^M \approx \frac{2n}{\alpha^F(n + \gamma^F - r)}. \quad (10)$$

Together with our previous approximations, this in turn gives a simple approximation for the probability of a male accepting in period r :

$$\frac{s_r^M}{r} \approx \frac{R_r^M}{n} \approx \frac{2}{\alpha^F(n + \gamma^F - r)}. \quad (11)$$

5.1 Accuracy of approximation

The accuracy of this approximative solution is remarkably good, as is evident in Fig. 1. Similar approximative solutions to related recurrences were analyzed by [Chow et al. \(1964\)](#) and [Eriksson et al. \(2006\)](#), in which cases the relative error was proved to be at most on the order of $n^{-1/6}$. Since these proofs were very technical, we here refrain from proving similar error bounds, relying on the evidence that the approximation is good.

We can check the validity of the approximation in two special cases of interest. When α^F tends to zero, males must take every chance to try to mate any female that is better than $u^M n$, and indeed, in (11) we obtain $s_r^M/r \rightarrow u^M$, regardless of period r .

The case when $\alpha^F = 1$ and $u^M = 1$ is the problem studied by [Chow et al. \(1964\)](#). They show that for large values of n , the critical period when s_r^M changes value from 0 to 1 is at $r_{\text{crit}} \approx n/3.87$. Our approximation (11) implies the value $r_{\text{crit}} \approx n/3$ for this case, which is not exactly right but quite close.

5.2 Approximation of α_{opt}^M

We shall now derive an approximation of α_{opt}^M , the optimal acceptance rate for males given their beliefs about the female acceptance rate. According to (11), the expression (8) for p_r^M has a simple approximation:

$$\begin{aligned}
 p_r^M &\approx \prod_{i=1}^{r-1} \left(1 - \frac{2}{n + \gamma^F - i} \right) = \prod_{i=1}^{r-1} \frac{n + \gamma^F - i - 2}{n + \gamma^F - i} \\
 &= \frac{(n + \gamma^F - r - 1)(n + \gamma^F - r - 2)}{(n + \gamma^F - 1)(n + \gamma^F - 2)} \\
 &\approx \frac{(n + \gamma^F - r)^2}{(n + \gamma^F)^2}.
 \end{aligned}$$

Following Chow et al. (1964), let r_{crit} be the minimal r such that $s_r^M > 0$, i.e. the first period in which males ever accept females. Equations (6) and (7) together with our approximation of p_r^M yield

$$\alpha_{\text{opt}}^M \approx \frac{(2/\alpha^F) \sum_{r=r_{\text{crit}}}^n (n + \gamma^F - r)}{\sum_{r=1}^n (n + \gamma^F - r)^2}.$$

where we have used the fact that $s_i^M = 0$ for $i < r_{\text{crit}}$. From (11) we get $r_{\text{crit}} \approx \alpha^F(n + \gamma^F - r_{\text{crit}})/2$. Hence, for large n we have $r_{\text{crit}} \approx n\alpha^F/(2 + \alpha^F)$. Using this approximation and applying standard formulas for arithmetic and quadratic series, we finally obtain a fundamental equality governing equilibria in this model.

$$\alpha_{\text{opt}}^M \approx g(\alpha^F, u^M) \stackrel{\text{def}}{=} \frac{3 \frac{2}{\alpha^F+2} \left(\frac{2}{\alpha^F+2} n + \frac{4}{u^M \alpha^F} \right)}{\alpha^F \left(n^2 + 3n \frac{2}{u^M \alpha^F} + 3 \left(\frac{2}{u^M \alpha^F} \right)^2 \right)}. \tag{12}$$

Figure 2b shows how this equation, and the corresponding equation where F and M are interchanged, approximate the exact best-response curves.

Lemma 1 *The function $g(\alpha^F, u^M)$ is strictly decreasing in α^F .*

Proof MAPLE computes the derivative (omitted here) which is easily seen to be negative.

Lemma 2 *The equation $\alpha = g(\alpha, u)$ has a unique root $\alpha = \sqrt{3/n} + o(1/\sqrt{n})$.*

Proof MAPLE computes

$$\lim_{n \rightarrow \infty} \frac{g(c/\sqrt{n}, u)}{c/\sqrt{n}} - 1 \stackrel{\text{def}}{=} h(c) = \frac{3 - c^2}{c^2}$$

which changes sign at $c = \sqrt{3}$. By Lemma 3 the equation $\alpha = g(\alpha, u)$ has a root $\alpha = \sqrt{3/n} + o(1/\sqrt{n})$. Uniqueness follows from Lemma 1.

5.3 The canonical equilibria

For an equilibrium (α^F, α^M) we have $\alpha^M \approx g(\alpha^F, u^M)$ and $\alpha^F \approx g(\alpha^M, u^F)$. Thus we ask ourselves: what are the roots α^F to the following equation?

$$g(g(\alpha^F, u^M), u^F) = \alpha^F \tag{13}$$

Unfortunately this is a polynomial equation of degree 17 so in general we cannot solve it, but with a computer program for symbolic calculations, like MAPLE, we are able to say something about the roots when n tends to infinity.

Let α_{\max}^F be the greatest $\alpha^F \in (0, 1)$ that is a root to (13). Modulo our approximations, this is the greatest α^F such that (α^F, α^M) is an equilibrium for some α^M . Analogously, we define α_{\min}^F as the smallest $\alpha^F \in (0, 1)$ that is a root to (13).

To state our results we need the following definition.

Definition 1 Let $f(n), g(n)$ be two functions.

- We write $f < g$ if for each $\varepsilon > 0$ we have $(1 - \varepsilon)f(n) < g(n)$ for sufficiently large n .
- We write $f(n) = o(g(n))$ if $f(n)/g(n) \rightarrow 0$ when $n \rightarrow \infty$.
- We write $f(n) = \Theta(g(n))$ if $f(n) \neq o(g(n))$ and there is a positive constant C such that, for all sufficiently large n , $|f(n)/g(n)| < C$.

Theorem 1 If $\alpha^F \in [\alpha_{\min}^F, \alpha_{\max}^F]$ we have $2.18/n < \alpha^F g(\alpha^F, u^M) < 3/n$ for large n .

We omit the proof which is a technical exercise in MAPLE.

Theorem 2 For sufficiently large n we have

$$c_0/n < \alpha_{\min}^F < \alpha_{\max}^F < \max\{0, u^F - 2/3\} + 0.008,$$

where $c_0 = 632812500 / ((375u^M + 503)^2(375u^M - 247))$. Furthermore, if $u^F > 2/3$ we have

$$|\alpha_{\max}^F - (u^F - 2/3)| < 0.008$$

for large n .

Theorem 3 There is a positive root $\alpha^F = \Theta(1/\sqrt{n})$ to (13) if and only if either

- $u^F = u^M = 2/3$ in which case $\alpha^F = \sqrt{3/n} + o(1/\sqrt{n})$, or
- $(u^F - 2/3)(u^M - 2/3) > 0$ in which case

$$\alpha^F = \sqrt{\frac{3}{n} \cdot \frac{u^M - 2/3}{u^F - 2/3} \cdot \frac{u^F}{u^M}} + o(1/\sqrt{n}).$$

For the proofs we refer to the Appendix.

5.4 Interpretation of the main theorems

Going back to the computational analysis, we can now see how the observed behavior of equilibria is explained by the above theorems.

First, Theorem 1 explains the smooth look of the product $\alpha^F \alpha^M$ in equilibrium (Fig. 4). In essence, whichever equilibrium we happen to be in, the product of the acceptance rates is a constant, $3/n$. Thus, one sex can lower its acceptance rate only if the other sex raises its rate.

Second, Theorem 2 explains why the maximal acceptance rate tends to a constant as n grows for fixed u^F and u^M (Fig. 4), and why, for a fixed large n , the maximal acceptance rate grows linearly with u^F for $u^F > 2/3$ (Fig. 5).

Third, for $u^F = u^M < 2/3$, Theorem 3 explains the acceptance rates $\alpha^F = \alpha^M = \sqrt{3/n}$ in the essentially unique equilibrium (Fig. 5). It also explains the asymmetry of this equilibrium when $u^F > u^M$ (Fig. 6).

Finally, Theorem 2 explains why, in the asymmetric case $u^F > u^M$, there suddenly appears an equilibrium favoring females when $u^M > 2/3$ (Fig. 6).

5.5 The advantage of being choosy

Our analysis has been focused on the acceptance rates in equilibrium. A lower acceptance rate says that the agents of this sex are ‘choosy’. We are now in a position to show that it is advantageous to be part of a choosy sex in terms of obtaining a mate of good expected rank. According to Eq. (10), the expected rank of mates for males is, essentially, inversely proportional to α^F . According to Theorem 1, in equilibrium the male acceptance rate is also inversely proportional to α^F . Consequently, the expected rank of males’ mates is roughly proportional to the male acceptance rate. In other words, an equilibrium where the males are choosy compared to the females, is an equilibrium where the males end up with on average better mates.

5.6 Comparison with a model where there is only one cohort of agents

In the symmetric case $u^M = u^F = u$, a symmetric equilibrium would have $\alpha^M = \alpha^F \approx \sqrt{3/n}$ when u and n are large, according to Theorem 1. Plugging this into (10) yields $R_1^M \approx \frac{2}{\sqrt{3}}\sqrt{n}$.

Eriksson et al. (2006) studied the same symmetric model with the exception that agents could only date other agents of their own cohort. In that model, agents entering the game could expect to obtain a partner of rank \sqrt{n} . The differing factor of $2/\sqrt{3}$ is quite small, which indicates that expected quality of mates is not particularly sensitive to the possibility of mating across age-groups.

6 Discussion

We have analyzed a game-theoretic model of mutual mate choice. The three main features of this model, as compared to previous models such as those proposed by

Johnstone (1997) or Alpern and Reyniers (2005), are our assumptions that agents have independent preferences, that they are trying to optimize rank relative to these preferences (rather than some marker of absolute fitness), and that agents can mate across cohorts. In this model, we find that even in symmetric games, there exist asymmetric equilibria. We also find that the set of equilibria collapses to a unique “canonical” equilibrium when at least one sex prefers no mate at all to a sufficient proportion of bad mates.

Asymmetric equilibria of symmetric games have not been observed in previous models (Alpern and Reyniers 2005). We do not think that the general key to the phenomenon is whether mating across cohorts is allowed; indeed, we have carried out computations on such a model with common preferences, and no asymmetric equilibria emerged. Instead, it seems to us that the key to asymmetric equilibria lies in preferences being sufficiently independent. As we mentioned in the introduction, we got this idea from the theory of stable matching. Eriksson and Hågström (2007) have found that even decentralized matching tend to lead to near-stable outcomes. When preferences are common, there exists a unique stable matching, namely the perfectly assortative matching. When preferences are not common, there usually exist several stable matchings, one of which is optimal for all males (and worst possible for all females).

Under our assumption of perfectly independent preferences, the model became amenable to mathematical analysis. To begin with, since there is no consensus on who is most attractive, all agents of the same sex can be treated equally in the model. Furthermore, there is no issue of whether agents know their own attractiveness from the beginning or learn it over time. Finally, we need only assume that agents can rank-order potential mates that they have seen so far; there is no issue of learning the range of attractiveness of the other sex. These simplifications made it possible for us to obtain approximative analytic solutions to otherwise very complex games. Arguably, independent preferences is far from any real application, but they represent a null hypothesis about correlations of preferences which can serve as a benchmark. Anyway, true mating preferences will certainly deviate from perfectly common preferences. Apart from everyday experience of different tastes in partners, there is a biological reason. As discussed in a comprehensive review by Kokko et al. (2003), mate choice mechanisms evolve and hence variation in mating preferences are to be expected. Indeed, Bergstrom and Real (2000) claim that “variation in mating preferences may be a ubiquitous outcome of genetic and environmental heterogeneity”.

Thus, it seems likely that sex differences in mating strategies are determined not only by exogenously asymmetric factors like sex ratio or choice costs (Johnstone 1997), but can emerge also by nature’s selection among multiple equilibria.

An obvious prediction of our model is that we should expect to find a gender difference in how preferable men and women find their spouses relative other potential partners.

Acknowledgments This work was partially supported by a grant to the first author from the Swedish Research Council. We are very grateful to an anonymous reviewer for many valuable suggestions.

Appendix: Proof of Theorems 2 and 3

Theorems 2 and 3 state important properties of the limit as $n \rightarrow \infty$ of the acceptance rate α^F . In our approximative model α^F can be found as a root of Eq. (13), a polynomial equation of degree 17. In order to get a handle on these roots as n tends to infinity we first need a general lemma about root functions.

A function $a(n)$ is called a *root function* to a sequence $(f_n)_{n=1}^\infty$ of functions if $f_n(a(n)) = 0$ for all sufficiently large n . We say that a real function $h(c)$ *changes sign* at $c = c_0$ if in some neighborhood of c_0 we have $h(c) < 0$ if $c < c_0$ and $h(c) > 0$ if $c > c_0$ or the other way around.

Lemma 3 *Let $(f_n(x))_{n=1}^\infty$ be a sequence of continuous real functions. Let $F(n)$ be a positive real function, let β be any real number and suppose $h(c) = \lim_{n \rightarrow \infty} f_n(cF(n))n^\beta$ exists or is infinite for all $c > 0$. Then the following holds.*

1. *If $h(c)$ is a continuous function for all $c > 0$, then there is a positive root function $a(n) = \Theta(F(n))$ to (f_n) only if $h(c_0) = 0$ for some $c_0 > 0$.*
2. *If $h(c)$ changes sign at $c = c_0 > 0$, then there is a root function $a(n)$ to (f_n) such that $a(n) = c_0F(n) + o(F(n))$.*

Proof 1. Suppose $a(n) \in \Theta(F(n))$ is a positive root function to (f_n) . Since $\lim_{n \rightarrow \infty} a(n)/F(n) \neq 0$ there is a constant $B > 0$ such that $a(n)/F(n) > B$ for infinitely many n . There is also a constant $C > 0$ such that $a(n)/F(n) < C$ for all sufficiently large n . By Bolzano-Weierstrass's theorem there is an accumulation point c_0 of the set $S = (B, C) \cap \{a(n)/F(n) : n = 1, 2, \dots\}$.

We will show that $h(c_0) = 0$. Suppose not. Then by continuity of $h(c)$ we can choose $\varepsilon > 0$ such that $|h(c) - h(c_0)| < |h(c_0)|$ whenever $|c - c_0| < \varepsilon$. But since c_0 is an accumulation point of S , there is an $s \in S$ such that $|s - c_0| < \varepsilon$. Now $|h(s) - h(c_0)| = |h(c_0)|$ gives a contradiction.

2. Put $h_n(c) = f_n(cF(n))n^\beta$ so that $h(c) = \lim_{n \rightarrow \infty} h_n(c)$.

Assume for simplicity that $h(c)$ changes sign from minus to plus at $c = c_0$. (The from-plus-to-minus case is completely analogous.) Then there is an $\hat{\varepsilon} > 0$ such that $h(c_0 - \varepsilon) < 0 < h(c_0 + \varepsilon)$ for every $0 < \varepsilon < \hat{\varepsilon}$. This means that for each $0 < \varepsilon < \hat{\varepsilon}$ there is an $n(\varepsilon)$ such that $h_n(c_0 - \varepsilon) < 0 < h_n(c_0 + \varepsilon)$ whenever $n > n(\varepsilon)$. Let us choose the function $n(\varepsilon)$ so that $n(\varepsilon) \rightarrow \infty$ when $\varepsilon \rightarrow 0$.

Now define a function $\varepsilon(n)$ by

$$\varepsilon(n) = \hat{\varepsilon} / (\min\{i \in \{2, 3, \dots\} : n(\hat{\varepsilon}/i) \geq n\} - 1).$$

The set $T = \{i \in \{2, 3, \dots\} : n(\hat{\varepsilon}/i) \geq n\}$ is nonempty since $n(\varepsilon) \rightarrow \infty$ when $\varepsilon \rightarrow 0$. If $\min T \geq 3$ clearly $n(\varepsilon(n)) < n$ and if $\min T = 2$ we have $\varepsilon(n) = \hat{\varepsilon}$. Thus, for any $n > n(\hat{\varepsilon})$ we have $n > n(\varepsilon(n))$ and hence $h_n(c_0 - \varepsilon(n)) < 0 < h_n(c_0 + \varepsilon(n))$. Since f_n is continuous there is an $a(n)$ with $(c_0 - \varepsilon(n))F(n) < a(n) < (c_0 + \varepsilon(n))F(n)$ such that $f_n(a(n)) = 0$. It follows directly from the definition of $\varepsilon(n)$ that $\varepsilon(n) \rightarrow 0$ when $n \rightarrow \infty$, and therefore $a(n) = c_0F(n) + o(F(n))$.

Now we are ready to prove Theorem 3.

Proof First assume that $u^F = u^M = 2/3$. Thanks to MAPLE we get

$$\lim_{n \rightarrow \infty} \left(\frac{g(g(c/\sqrt{n}, 2/3), 2/3)}{c/\sqrt{n}} - 1 \right) n = h(c) \stackrel{\text{def}}{=} \frac{63 - 7c^4}{4c^2}.$$

Clearly $h(c)$ vanishes only at $c = \sqrt{3}$ where it changes sign, so Lemma 3 gives the desired result.

Now assume instead that $u^F \neq 2/3$ or $u^M \neq 2/3$. MAPLE computes

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{g(g(c/\sqrt{n}, u^M), u^F)}{c/\sqrt{n}} - 1 \right) \sqrt{n} \\ &= H(c) \stackrel{\text{def}}{=} \frac{(u^F - 2/3)u^M c^2 - 3(u^M - 2/3)u^F}{u^F u^M c}. \end{aligned}$$

which is continuous in c and vanishes if and only if

$$c = c_0 \stackrel{\text{def}}{=} \sqrt{3 \cdot \frac{u^M - 2/3}{u^F - 2/3} \cdot \frac{u^F}{u^M}}$$

which is well-defined if and only if $(u^F - 2/3)(u^M - 2/3) > 0$. Since $H(c)$ has the same sign as $(c^2 - c_0^2)(u^F - 2/3)$ it changes sign at $c = c_0$, and again Lemma 3 does the job.

To prove Theorem 2 we need one more lemma, which basically says that if the acceptance rate α^F is asymptotically constant, then this constant must be close to $u^F - 2/3$ (in which case of course u^F must be greater than $2/3$).

Lemma 4 *There is a root $0 < \alpha^F \leq 1$ to (13) with $\alpha^F = \Theta(1)$ if and only if $u^F > 2/3$, in which case $\alpha^F = A(u^F) + o(1)$ where $A(u^F)$ is some function satisfying $|A(u^F) - (u^F - 2/3)| < 0.008$.*

Proof Put

$$h(\alpha^F, u^F) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} g(g(\alpha^F, u^M), u^F) / \alpha^F - 1.$$

MAPLE easily computes

$$\begin{aligned} & h(\alpha, u^F) \\ &= \frac{(\alpha + 2)^2(\alpha^3 + 4\alpha^2 + 4\alpha + 3u^F)u^F}{12(u^F)^2 + \alpha^6 + 8\alpha^5 + 24\alpha^4 + 32\alpha^3 + 16\alpha^2 + 6u^F\alpha^3 + 24\alpha^2u^F + 24\alpha u^F} - 1 \end{aligned}$$

which is continuous, and if this should be equal to 0 we must have

$$u^F = U(\alpha) \stackrel{\text{def}}{=} \frac{(\sqrt{\alpha^4 + 8\alpha^3 + 24\alpha^2 + 32\alpha + 4} - \alpha^2 - 4\alpha + 2)(\alpha + 2)^2}{6(\alpha + 4)}.$$

By some standard calculus one can check that $U'(\alpha) > 0$ for all $0 \leq \alpha \leq 1$. Since $U(0) = 2/3$ and $U(1) > 1$ we have proved the “only if” part of the lemma using part (1) of Lemma 3.

If we let MAPLE compute the partial derivative $\frac{\partial}{\partial u^F} h(\alpha, U(\alpha))$ the result (which we omit here) is an expression (only dependent on α) which is easily seen to be positive whenever $\alpha > 0$. Since $h(\alpha, U(\alpha)) = 0$ we have

$$0 = \frac{d}{d\alpha} h(\alpha, U(\alpha)) = \left(\frac{\partial}{\partial \alpha^F} + U'(\alpha) \frac{\partial}{\partial u^F} \right) h(\alpha, U(\alpha))$$

and we conclude that

$$\frac{\partial}{\partial \alpha^F} h(\alpha, U(\alpha)) < 0$$

whenever $\alpha > 0$.

Since $U(\alpha)$ is a strictly increasing function on the interval $[0, 1]$, it has a strictly increasing inverse $A(u)$ defined on the interval $[U(0), 1] = [2/3, 1]$. We can now reformulate our result in the last paragraph:

$$\frac{\partial}{\partial \alpha^F} h(A(u^F), u^F) < 0,$$

that is, as we change α^F , $h(\alpha^F, u^F)$ changes sign at $\alpha^F = A(u^F)$ if $u^F > 2/3$. Part (2) of Lemma 3 now gives the desired result.

A plot (or a Taylor series) reveals that $U(\alpha^F) \approx \alpha^F + 2/3$ and by some standard calculus (omitted here) it is possible to conclude that $|A(u^F) - (u^F - 2/3)| < 0.008$.

Finally, we are ready for the proof of Theorem 2:

Proof The inequality $\alpha_{\max}^F < \max\{0, u^F - 2/3\} + 0.008$ and the implication

$$u^F > 2/3 \Rightarrow |\alpha_{\max}^F - (u^F - 2/3)| < 0.008$$

follows directly from Lemma 4 for large n . It remains only to prove $c_0/n < \alpha_{\min}^F$. Lemma 1 yields

$$\alpha_{\min}^F = g(\alpha_{\max}^M, u^F) > g(u^M - 2/3 + 0.008, u^F).$$

MAPLE tells us that

$$n \cdot g(u^M - 2/3 + 0.008, u^F) \rightarrow c_0$$

when $n \rightarrow \infty$.

References

- Alpern S, Katrantzi I, Reyniers D (2005) Mathematical models of mutual mate choice. In: Recent research developments in experimental & theoretical biology, vol. 1. World Scientific, Singapore, pp 177–200
- Alpern S, Reyniers D (1999) Strategic mating with homotypic preferences. *J Theor Biol* 198:71–88
- Alpern S, Reyniers D (2005) Strategic mating with common preferences. *J Theor Biol* 237:337–354
- Bergstrom CT, Real LR (2000) Towards a theory of mutual mate choice: lessons from two-sided matching. *Evol Ecol Res* 2:493–508
- Chow YS, Moriguti S, Robbins H, Samuels SM (1964) Optimal selection based on relative rank (the “Secretary problem”). *Isr J Math* 2:81–90
- Coombs R (1991) Marital status and personal well-being: a literature review. *Fam Relat* 40:97–102
- Demange G, Gale D (1985) The strategy structure of two-sided matching markets. *Econometrica* 53(4): 873–888
- Eriksson K, Häggström O (2007) Instability of matchings in decentralized markets with various preference structures. *Int J Game Theory* (this issue)
- Eriksson K, Sjöstrand J, Strimling P (2006) Optimal expected rank in a two-sided secretary problem. *Oper Res* (to appear)
- Gale D, Shapley L (1962) College admissions and the stability of marriage. *Am Math Mon* 69:9–15
- Geary D (2000) Evolution and proximate expression of human paternal investment. *Psychol Bull* 126:55–77
- Geary D (2005) Evolution of paternal investment. In: Buss DM, (ed) *The evolutionary psychology handbook*. Wiley, New York, pp 483–505
- Johnstone R (1997) The tactics of mutual mate choice and competitive search. *Behav Ecol Sociobiol* 40:51–59
- Johnstone RA, Reynolds JD, Deutsch JC (1996) Mutual mate choice and sex differences in choosiness. *Evolution* 50:1382–1391
- Knuth DE (1976) *Mariages Stables*. Les Presses de l’Université de Montréal
- Kokko H, Brooks R, Jennions MD, Morley JM (2003) The evolution of mate choice and mating biases. *Proc R Soc Lond B* 270:653–664
- Parker GA (1983) Mate quality and mating decisions. In: Bateson P (ed) *Mate choice*. Cambridge University Press, Cambridge, pp 141–164
- Ramsey D, Szajowski K (2005) Bilateral approach to the secretary problem. In: Nowak AS, Szajowski K (eds) *Advances in dynamic games: applications to economics, finance, optimization, and stochastic control*. *Annals of the International Society of Dynamic Games*, vol 7, pp 271–284
- Real LA (1991) Search theory and mate choice: II. Mutual interaction, assortative mating, and equilibrium variation in male and female fitness. *Am Nat* 138:901–917
- Roth AE, Sotomayor MAO (1990) *Two-sided matching*. Cambridge University Press, Cambridge
- Sassler S, Schoen R (1999) The effect of attitudes and economic activity on marriage. *J Marriage Fam* 61(1):147–159
- Stack S, Eshleman JR (1998) Marital status and happiness: a 17-nation study. *J Marriage Fam* 60(2):527–536
- Wood W, Rhodes N, Whelan M (1989) Sex differences in positive well-being: a consideration of emotional style and marital status. *Psychol Bull* 106:249–264